CS4641B Machine Learning

Focus video: SVD

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Singular value decomposition

- $\mathbf{X}_{N\times D}$, *N* is the number of dataset instances, *D* is the dimensionality of each instance (i.e. the number of features) and $\mathbf{X}$ is a centered matrix
- The singular value decomposition is given by

$$\mathbf{X} = \mathbf{U\Sigma V}^{\mathrm{T}} \rightarrow \text{FACTORS}$$

Where

- $\mathbf{U}_{N\times N} \rightarrow \textit{unitary matrix} \rightarrow \mathbf{UU}^{\mathrm{T}} = \mathbf{I}$
- $\mathbf{\Sigma}_{N\times D} \rightarrow \textit{diagonal matrix}$
- $\mathbf{V}_{D\times D} \rightarrow \textit{unitary matrix} \rightarrow \mathbf{VV}^{\mathrm{T}} = \mathbf{I}$

Singular value decomposition

$$\mathbf{X} = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{\mathbf{V}^{\mathrm{T}}}$$

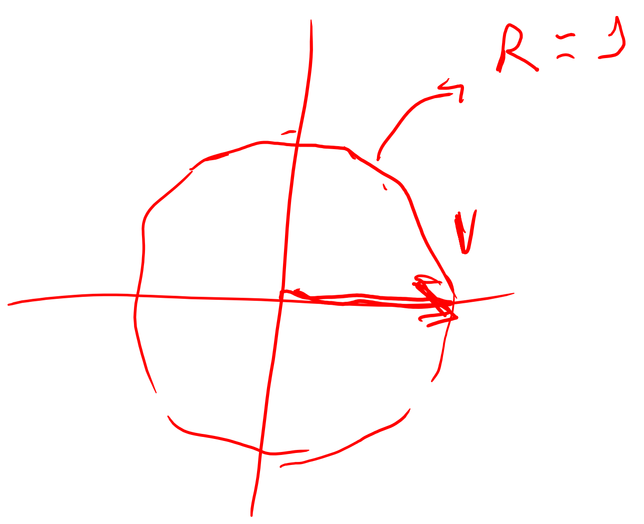
How to compute this singular value decomposition

Geometric meaning of SVD

Apply linear transformation to a vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\|v\|_2 = 1$

$$\mathbf{y} = \mathbf{X}\mathbf{v} = \underbrace{\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}}_{x} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{y}$$



Replacing with the SVD of $\mathbf{X}$:

$$\mathbf{y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Geometric meaning of SVD

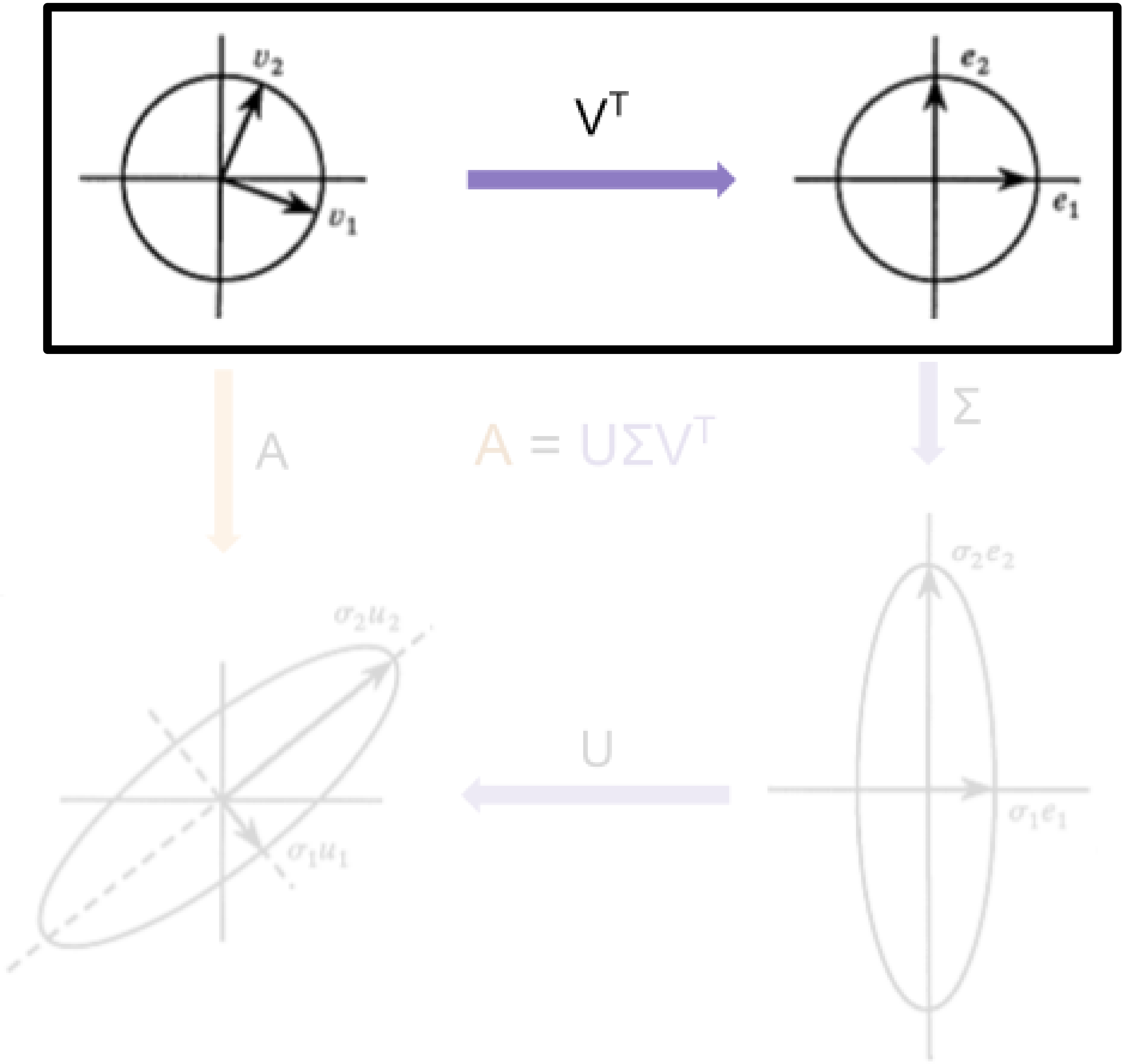


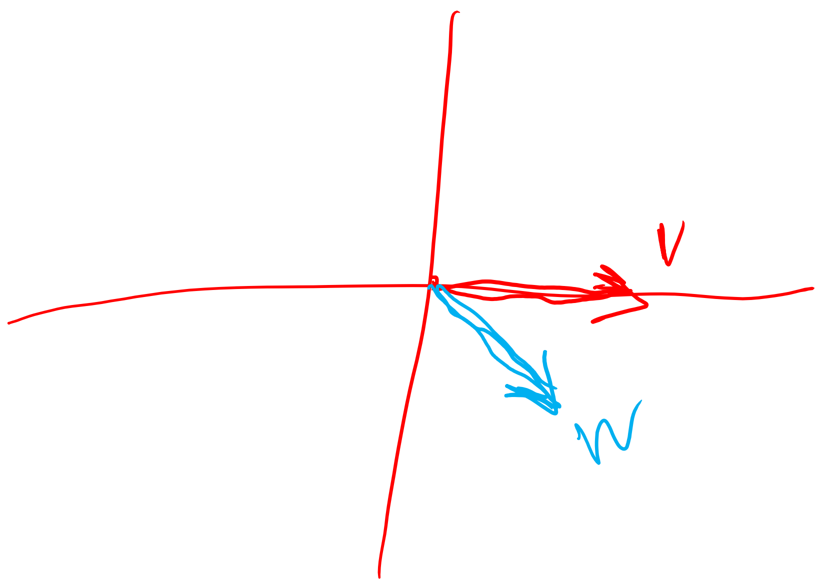
Image credit: Kevin Binz

Geometric meaning of SVD

$$\mathbf{w} = \mathbf{V}^{\mathrm{T}}\mathbf{v} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}}_{w}$$

$$\|w\|_2 = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(-\tfrac{1}{\sqrt{2}}\right)^2}$$

$$\|w\|_2 = 1$$



ROTATION

Geometric meaning of SVD

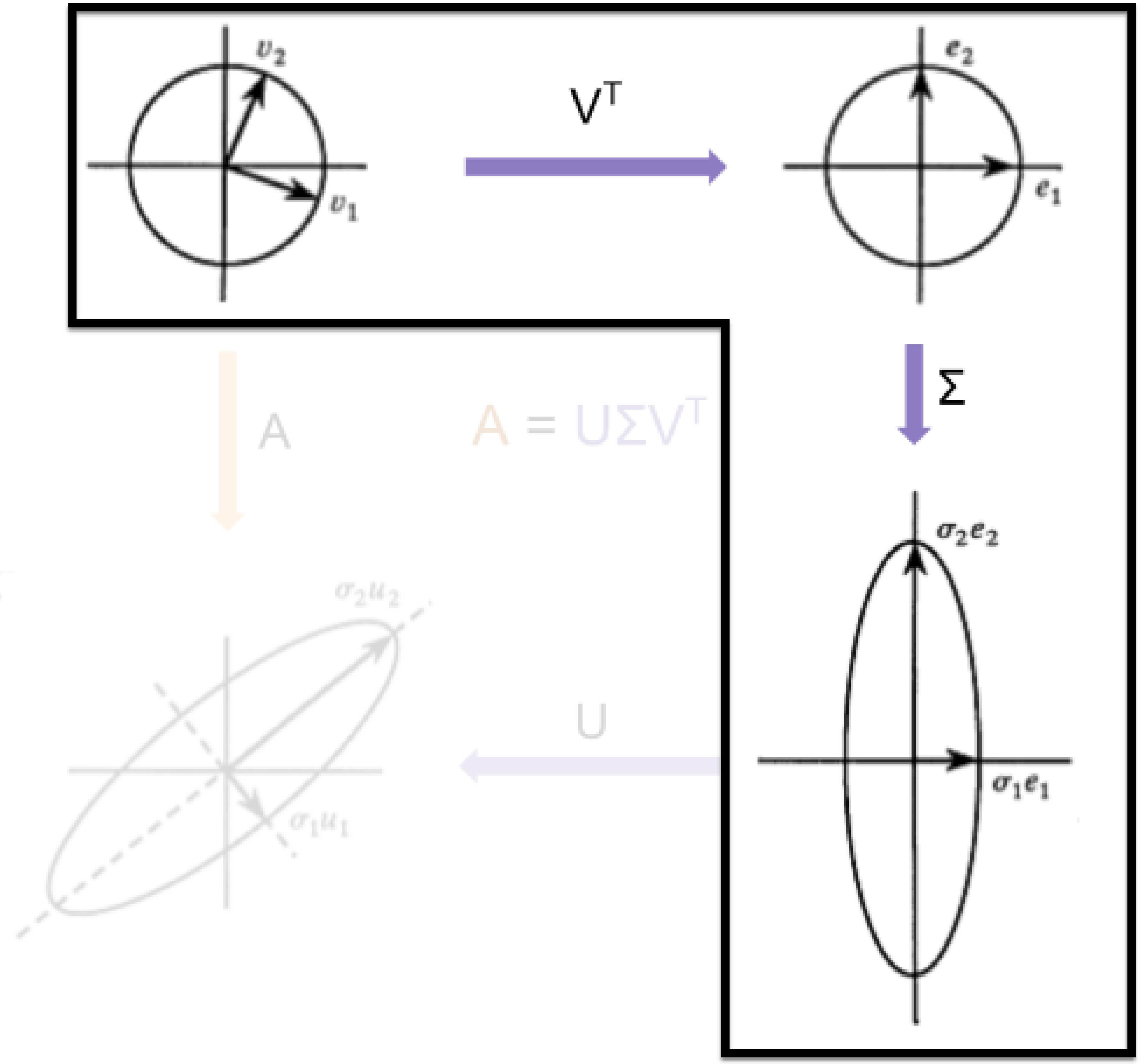


Image credit: Kevin Binz

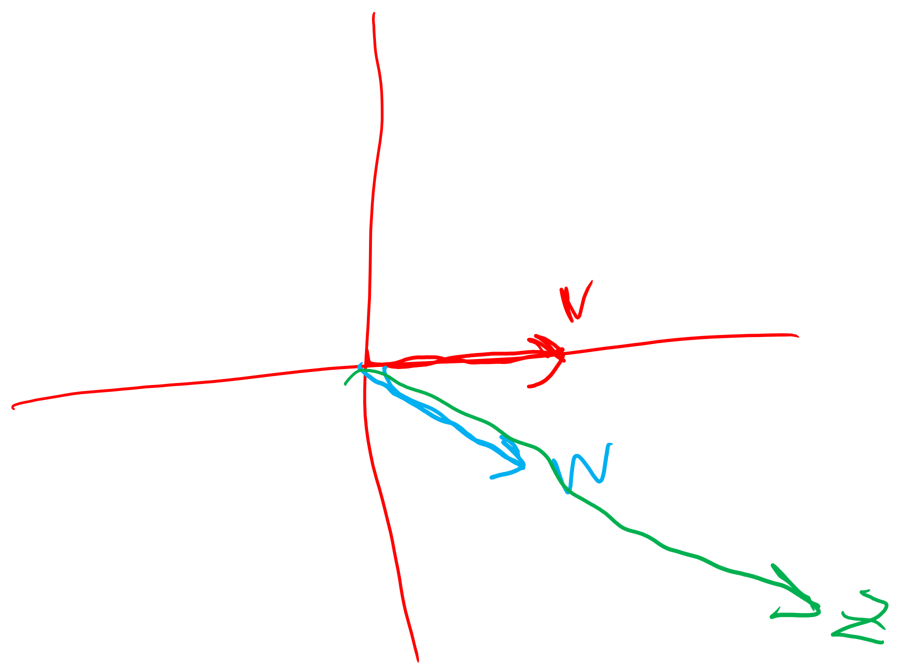
Geometric meaning of SVD

$$\mathbf{z} = \boldsymbol{\Sigma}\mathbf{w} = \underbrace{\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}}_{w} = \begin{bmatrix} \frac{\sqrt{45}}{\sqrt{2}} \\ -\frac{\sqrt{5}}{\sqrt{2}} \end{bmatrix}$$

$$\|z\|_2 = \sqrt{\frac{45}{2} + \frac{5}{2}}$$

$$\|z\|_2 = 5$$

SCALING



Geometric meaning of SVD

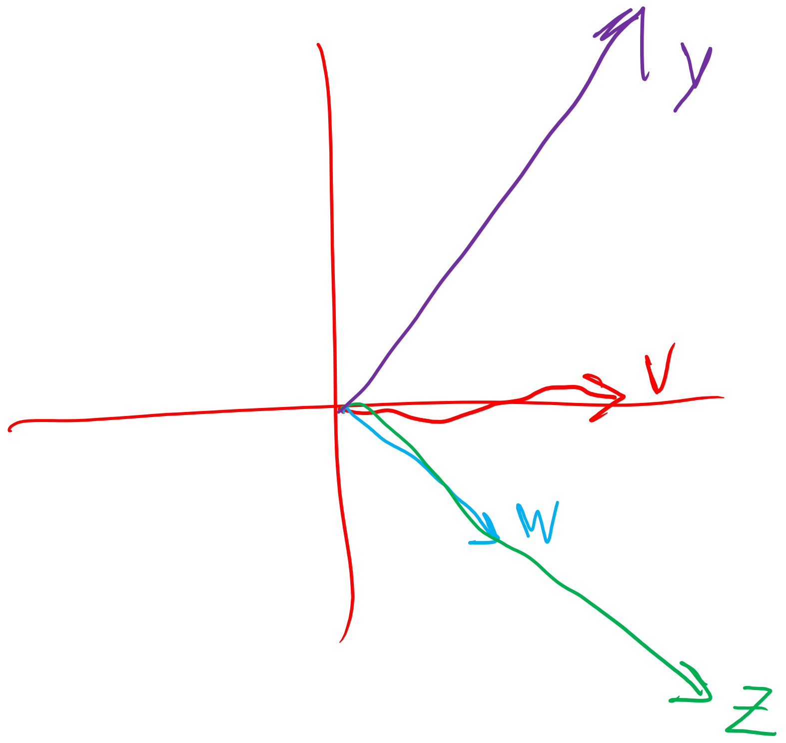
$$\mathbf{y} = \mathbf{Uz} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \frac{\sqrt{45}}{\sqrt{2}} \\ -\frac{\sqrt{5}}{\sqrt{2}} \end{bmatrix}}_{z} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \rightarrow x_v$$

$$\|y\|_2 = \sqrt{3^2 + 4^2}$$

$$\|y\|_2 = 5$$

$$\mathbf{y} \rightarrow x_v = U\Sigma V^T v$$

ROTATION



Geometric meaning of SVD

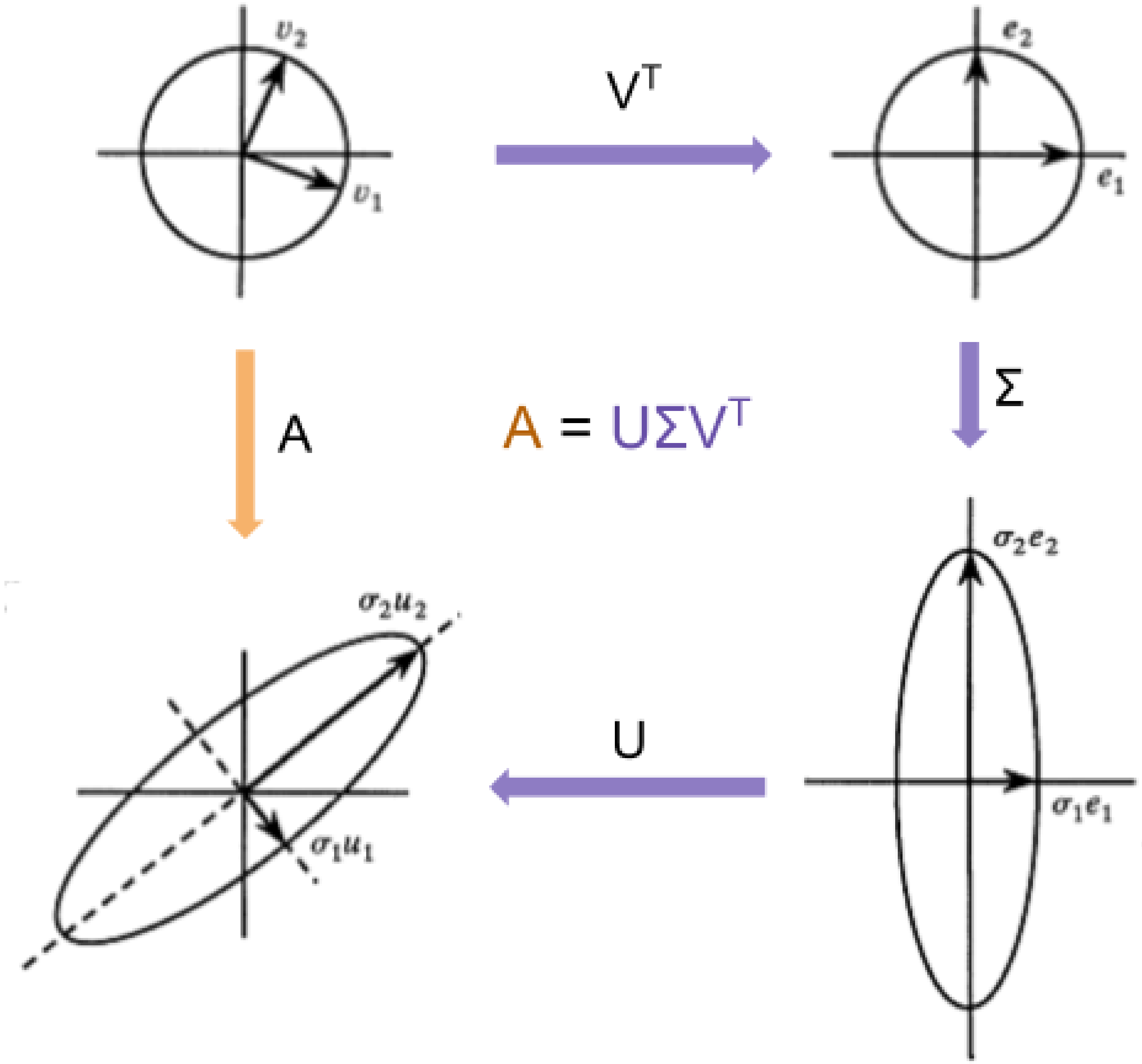


Image credit: Kevin Binz